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SURVEY

ON GENERALIZED METRIC SPACES: A SURVEY

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Abstract. We present a survey of fixed point results in generalized metric spaces (g.m.s.) in the sense of Branciari [Branciari, A., (2000), A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57, 31–37]. Since it may happen that the topology of such space is not Hausdorff, several authors added Hausdorfness (or some other condition) as an additional assumption in order to obtain their results. We show here that such assumptions are usually superfluous. Finally, we state some open questions on the topic.

Keywords: generalized metric space; quadrilateral inequality; fixed point; Hausdorff space.

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1. Introduction

Metric spaces form a natural environment for exploring fixed points of single and multivalued mappings. Metric fixed point theory has been an area of vigorous scientific activity since the basic result of Banach in 1922. Applications of such results cover several areas of mathematics and other sciences.

It may be noted that the use of triangle inequality in metric arguments is of extreme importance since it implies, among other things, the following:

(1) The metric d is continuous in both variables.
(2) The respective topology is Hausdorff.
(3) In particular, a sequence may converge to at most one point.
(4) Each open ball is an open set.
(5) Each convergent sequence is a Cauchy sequence.

Due to the nature of mathematics science, there has been many attempts to generalize the metric setting by modifying some of the axioms of metric spaces. Thus, several other types of spaces has been introduced and a lot of metric results has been extended to new settings.

One of the interesting generalizations of the notion of metric space was introduced by Branciari in 2000 [8], where the triangle inequality was replaced by a so-called rectangular (or quadrilateral) inequality, involving four (or more) instead of three points. It was not immediately observed that such spaces (called rectangular or generalized metric spaces, g.m.s., for short) may fail to satisfy conditions (1)–(5). Hence, in some of the first papers that followed, the authors implicitly used some of these conditions, so that their proofs were flawed.

Samet [35] and Sarma et al. [37] presented examples showing that there exist g.m. spaces that do not satisfy any of the properties (1)–(5). As a consequence, most of the authors dealing with

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such spaces made some additional requirements in order to deduce their results. However, we will show in this article that such requirements are usually superfluous, since some easy results, included those of Turinici [38], can be used to compensate the lack of (some of the) properties (1)–(5).

2. Generalized (rectangular) metric spaces

The following definition was given by Branciari in 2000.

Definition 2.1. [8] *Let X be a nonempty set, and let $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y,*

(i) $d(x, y) = 0$ *iff* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *("quadrilateral inequality")*

hold. Then (X, d) is called a generalized metric space (g.m.s., for short).

Thereafter, a great number of researchers deduced several (common) fixed point results in g.m. spaces, mostly extending known results from the setting of standard metric spaces (see [2-18, 20-32, 34-38]).

Convergent and Cauchy sequences in g.m.s., completeness, as well as open balls $B_r(p)$, are introduced in a standard way. For example, $B_r(p) = \{x \in X \mid d(p, x) < r\}$. However, their properties (1)–(5) (see the Introduction) may not hold. This was overlooked by some authors in the first papers concerning these spaces and, hence, the proofs of the corresponding fixed point results seemed not to be correct (see, e.g., [3, 6, 8, 12, 13, 28, 29]). Samet [35] and Sarma et al. [37] were the first to present examples showing this fact. We recall here the following

Example 2.1. [37, Example 1.1] Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define $d : X \times X \to [0, +\infty)$ as follows:

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B \\ y, & x \in A,\ y \in B \\ x, & x \in B,\ y \in A. \end{cases}$$

Then (X, d) is a complete g.m.s. However, it is easy to see that:

- the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ converges to both 0 and 2 and it is not a Cauchy sequence;
- there is no $r > 0$ such that $B_r(0) \cap B_r(2) = \emptyset$; hence, the respective topology is not Hausdorff;
- $B_{2/3}(\frac{1}{3}) = \{0, 2, \frac{1}{3}\}$, however there does not exist $r > 0$ such that $B_r(0) \subseteq B_{2/3}(\frac{1}{3})$;
- $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ but $\lim\limits_{n\to\infty} d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})$; hence d is not a continuous function.

This can be also interpreted by saying that the topology induced by a generalized metric may be not sequential (see [14, Note 1]).

Consequently, most of the authors that worked in g.m.s. afterwards, assumed some additional conditions, usually the Hausdorffness of the induced topology (see, e.g.[5, 7-11, 15, 16, 18, 30, 37]). As samples, we present some of their results.

Theorem 2.1. [37, Theorem 1.3] *Let (X, d) be a Hausdorff and complete g.m.s. and let $T : X \to X$ be a mapping such that for some $\lambda \in [0, 1)$ and all $x, y \in X$,*

$$d(Tx, Ty) \leq \lambda d(x, y)$$

holds. Then T has a unique fixed point.

Theorem 2.2. [5, Theorem 2.1] *Let (X, d) be a Hausdorff and complete g.m.s. Suppose that $T : X \to X$ is such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Tx) + d(y, Ty)) - \phi(d(x, Tx), d(y, Ty)),$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is continuous, and $\phi(a, b) = 0$ iff $a = b = 0$. Then T has a unique fixed point.

Theorem 2.3. [10, Theorem 2.3] *Let (X, d) be a Hausdorff and complete g.m.s, and let φ : $[0, +\infty) \to [0, +\infty)$ satisfies: (φ_1) $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$; (φ_2) $\liminf_{t_n \to t} \varphi(t_n) < t$ for all $t > 0$. Let $S, T, F, G : X \to X$ be such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \varphi(\max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\}).$$

Assume that $T(X) \subseteq F(X)$ and $S(X) \subseteq G(X)$ and the pairs $\{S, F\}$ and $\{T, G\}$ are compatible. If F or G is continuous, then S, T, F and G have a unique common fixed point in X.

3. Generalized metric spaces without Hausdorff property

As shown in Example 2.1, a sequence in a g.m.s. may have two limits. However, there is a special situation where this is not possible, and this can be useful in some proofs. The following lemma is a variant of [19, Lemma 1.10].

Lemma 3.1. *Let (X, d) be a g.m.s. and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then the sequence $\{x_n\}$ can converge to at most one point.*

Proof. Suppose, to the contrary, that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$ and $x \neq y$. Since x_m and x_n are distinct elements, as well as x and y, it is clear that there exists $\ell \in \mathbb{N}$ such that x and y are different from x_n for all $n > l$. For $m, n > \ell$, the rectangular inequality implies that

$$d(x, y) \leq d(x, x_m) + d(x_m, x_n) + d(x_n, y).$$

Taking the limit as $m, n \to \infty$, it follows that $d(x, y) = 0$, i.e., $x = y$. Contradiction. □

Also, if a sequence in an g.m.s. is both convergent and Cauchy, then pathologies as in Example 2.1 cannot happen, as shown by the following result due to Turinici.

Lemma 3.2. *(*[38, Proposition 1], [27, Propostion 3]*) Let (X, d) be a g.m.s. and let $\{x_n\}$ be a sequence in X which is both Cauchy and convergent. Then the limit x of $\{x_n\}$ is unique. Moreover, if $z \in X$ is arbitrary, then $\lim_{n\to\infty} d(x_n, z) = d(x, z)$.*

Finally, the proof of the following lemma is similar as in the standard metric case. As an illustration, we reproduce it here.

Lemma 3.3. [21, Lemma 2] *Let (X, d) be a g.m.s. and let $\{y_n\}$ be a sequence in X with distinct elements ($y_n \neq y_m$ for $n \neq m$). Suppose that $d(y_n, y_{n+1})$ and $d(y_n, y_{n+2})$ tend to 0 as $n \to \infty$ and that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε as $k \to \infty$:*

$$d(y_{m_k}, y_{n_k}), \quad d(y_{m_k}, y_{n_k+1}), \quad d(y_{m_k-1}, y_{n_k}), \quad d(y_{m_k-1}, y_{n_k+1}). \tag{1}$$

Proof. Since $\{y_n\}$ is not a Cauchy sequence, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$, $d(y_{m_k}, y_{n_k}) \geq \varepsilon$ and n_k is the smallest integer satisfying this inequality, i.e., $d(y_{m_k}, y_l) < \varepsilon$ for $m_k < l < n_k$.

Let us prove that the first of the sequences in (1) tends to ε as $k \to \infty$. Note that, by the assumption, $d(y_{m_k}, y_{m_k+1}) \to 0$ and $d(y_{m_k}, y_{m_k+2}) \to 0$ as $k \to \infty$. Hence, it is impossible that $n_k = m_k + 1$ or $n_k = m_k + 2$ (because in either of these cases it would be impossible to have $d(y_{m_k}, y_{n_k}) \geq \varepsilon$). Thus, we can apply the quadrilateral inequality to obtain

$$\begin{aligned}\varepsilon \leq d(y_{m_k}, y_{n_k}) &\leq d(y_{m_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) \leq \\ &\leq \varepsilon + d(y_{n_k-2}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) \to \varepsilon,\end{aligned}$$

as $k \to \infty$, implying that $d(y_{m_k}, y_{n_k}) \to \varepsilon$ as $k \to \infty$.

In order to prove that the second sequence in (1) tends to ε as $k \to \infty$, consider the following two quadrilateral inequalities:

$$\begin{aligned}d(y_{m_k}, y_{n_k+1}) &\leq d(y_{m_k}, y_{n_k}) + d(y_{n_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k+1})\\ d(y_{m_k}, y_{n_k}) &\leq d(y_{m_k}, y_{n_k+1}) + d(y_{n_k+1}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}),\end{aligned}$$

which, together with $d(y_{m_k}, y_{n_k}) \to \varepsilon$ imply that $d(y_{m_k}, y_{n_k+1}) \to \varepsilon$ as $k \to \infty$.

The proof for the other two sequences can be done in a similar way, using the following quadrilaterals:

$$(y_{m_k-1}, y_{n_k}, y_{n_k-2}, y_{m_k}) \quad \text{and} \quad (y_{m_k}, y_{n_k}, y_{m_k-1}, y_{m_k-2}),$$

resp.

$$(y_{m_k-1}, y_{n_k+1}, y_{n_k}, y_{m_k}) \quad \text{and} \quad (y_{m_k}, y_{n_k}, y_{m_k+1}, y_{n_k-1}).$$

□

Using the preceding lemmas, it is easy to show that all the mentioned results from [3, 6, 8, 12, 13, 28, 29] are in fact true as their proofs can be repaired. As a sample, consider the following

Theorem 3.1. [29, Theorem 1] *Let T be a quasicontraction (in the sense of Lj. Ćirić, see* [33, (24)]*) on a g.m.s. (X, d) which is T-orbitally complete. Then:*
a) T has a unique fixed point u in X;
b) $\lim_{n\to\infty} T^n x = u$ for each $x \in T$;
c) $d(T^n x, u) \leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}$ for all $n \in \mathbb{N}$.

Kikina and Kikina claim in [26] that the proof of this Theorem in [29] is wrong, since it is concluded that from

$$d(T^n x, T^m x) \leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}, \tag{2}$$

where $m > n$, $x \in X$ is arbitrary, and $0 \leq q < 1$, it follows that $\{T^n x\}$ converges (to some $u \in X$) and that

$$d(T^n x, u) \leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}. \tag{3}$$

Then, in [26], a rather complicated argument is presented as a substitute for this conclusion.

However, it is easy to show that this conclusion in [29] is actually correct. Namely, (2) implies that $\{T^n x\}$ is a Cauchy sequence. Since X is T-orbitally complete, $\{T^n x\}$ converges to some $u \in X$. By Lemma 3.2, this limit is unique and, moreover, replacing x_n and z in Lemma 3.2 by $T^m x$ and $T^n x$, respectively, and letting $m \to \infty$, (3) readily follows.

In a very similar way, it can be shown that all standard metric fixed point results listed in the well-known paper [33] by Rhoades can be easily extended to g.m.s., without additional assumptions. This also applies to all results of the mentioned papers [5, 7-11, 15, 16, 18, 29, 30, 37], including Theorems 2.1, 2.2 and 2.3.

Some new results are, also without additional assumptions, proved in [21] and [22]. This includes common fixed point results under Geraghty-type conditions, those using altering distance or admissible functions, as well as Meir-Keeler and Boyd-Wong-type results. We will prove here a result which is without proof stated in [21].

Theorem 3.2. [21, Theorem 2] *Let (X, d) be a g.m.s. and let $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If, for some altering distance function ψ and some $c \in [0, 1)$,*

$$\psi(d(fx, fy)) \leq c\psi(d(gx, gy)) \tag{4}$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence. If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Here, as usual, $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if:
(i) ψ is increasing and continuous,
(ii) $\psi(t) = 0$ iff $t = 0$.

Proof. We will prove first that f and g cannot have more than one point of coincidence. Suppose to the contrary that there exist $w_1, w_2 \in X$ such that $w_1 \neq w_2$, $w_1 = fu_1 = gu_1$ and $w_2 = fu_2 = gu_2$ for some $u_1, u_2 \in X$. Then (4) would imply that

$$\psi(d(w_1, w_2)) = \psi(d(fu_1, fu_2)) \leq c\psi(d(gu_1, gu_2)) < \psi(d(w_1, w_2)),$$

which is impossible.

In order to prove that f and g have a coincidence point, take an arbitrary $x_0 \in X$ and, using that $f(X) \subseteq g(X)$, choose sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \quad \text{for } n = 0, 1, 2, \dots$$

If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0+1} is a coincidence point of f and g, and y_{n_0+1} is their (unique) point of coincidence.

Suppose now that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. Then, using (4), we get that

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &= \psi(d(fx_n, fx_{n+1})) \leq c\psi(d(gx_n, gx_{n+1})) \\ &= c\psi(d(y_{n-1}, y_n)) < \psi(d(y_{n-1}, y_n)).\end{aligned}$$

Hence, $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of positive real numbers, tending to some $\delta \geq 0$. Suppose that $\delta > 0$. Then, since

$$\psi(d(y_n, y_{n+1})) \leq c\psi(d(y_{n-1}, y_n)),$$

taking the limit as $n \to \infty$, we get that $\psi(\delta) \leq c\psi(\delta)$. But this is possible only if $\delta = 0$, a contradiction. Hence,

$$d(y_{n-1}, y_n) \to 0 \text{ as } n \to \infty. \tag{5}$$

In a similar way, one can prove that

$$d(y_{n-2}, y_n) \to 0 \text{ as } n \to \infty. \tag{6}$$

Suppose now that $y_n = y_m$ for some $n > m$ (and hence, by the way y_n's are chosen, $y_{n+k} = y_{m+k}$ for $k \in \mathbb{N}$). Then, (4) implies that

$$\begin{aligned}\psi(d(y_m, y_{m+1})) = \psi(d(y_n, y_{n+1})) \leq c\psi(d(y_{n-1}, y_n)) < \psi(d(y_{n-1}, y_n)) \leq \cdots \\ \cdots \leq c\psi(d(y_m, y_{m+1})) < d(y_m, y_{m+1}),\end{aligned}$$

a contradiction. Thus, in what follows, we can assume that $y_n \neq y_m$ for $n \neq m$.

In order to prove that $\{y_n\}$ is a Cauchy sequence, suppose that it is not. Then, by Lemma 3.3, using (5) and (6), we conclude that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences (1) tend to ε as $k \to \infty$. Using (4) with $x = x_{m_k}$ and $y = x_{n_k+1}$, one obtains

$$\psi(d(y_{m_k}, y_{n_k+1})) \leq c\psi(d(y_{m_k-1}, y_{n_k})).$$

Letting $k \to \infty$, it follows that $\psi(\varepsilon) \leq c\psi(\varepsilon)$, a contradiction.

Suppose, e.g., that the subspace $g(X)$ is complete (the proof when $f(X)$ is complete is similar). Then $\{y_n\}$ is a Cauchy sequence, tending to some $y^* \in g(X)$, i.e., $y^* = gz$ for some $z \in X$. In order to prove that $fz = gz$, suppose that $fz \neq gz$. Then, by Lemma 3.1, it follows that y_n differs from both fz and gz for n sufficiently large. Hence, we can apply the rectangular inequality to obtain

$$\begin{aligned} d(fz, gz) &\leq d(fz, fx_n) + d(fx_n, fx_{n+1}) + d(fx_{n+1}, gz) \leq \\ &\leq c\psi(d(gz, gx_n)) + d(y_n, y_{n+1}) + d(y_{n+1}, gz) \to 0, \end{aligned}$$

as $n \to \infty$. It follows that $fz = gz$ is a point of coincidence of f and g.

In the case when f and g are weakly compatible, a well-known result implies that f and g have a unique common fixed point. □

4. Some additional results and open questions

4.1. **G.m.s. and Caristi's theorem.** Recently, in an interesting paper [27], Kirk and Shahzad proved that the well-known Caristi's theorem can also be proved in g.m.s. without additional assumptions. We just state here their main result, noting that the proof (as well as in the case of metric spaces) uses a kind of transfinite induction argument.

Theorem 4.1. [27, Theorem 2] *Let (X, d) be a complete g.m.s. Let $T : X \to X$ be a mapping, and let $\varphi : X \to [0, +\infty)$ be a lower semicontinuous function. Suppose that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad x \in X.$$

Then T has a fixed point.

4.2. **Fixed point results in compact g.m.s.** It is well-known that in compact metric spaces, fixed point results can be obtained under strict contractive conditions. In the case of g.m.s. with a continuous general metric, the following results of Nemytzki and Edelstein-type can be obtained in the same way as in the metric case.

Proposition 4.1. *Let (X, d) be a compact g.m.s. with continuous generalized metric d and let $f, g : X \to X$ be two self maps such that $f(X) \subset g(X)$, one of these two subsets of X being closed. Suppose that the following conditions hold:*

$$d(fx, fy) < d(gx, gy) \text{ for } gx \neq gy$$
$$\text{and } fx = fy \text{ whenever } gx = gy.$$

Then f and g have a unique point of coincidence say $y^* \in X$. Moreover, for each $x_0 \in X$, the corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = y^*$. In addition, if f and g are weakly compatible, then they have a unique common fixed point.

Proposition 4.2. *Let (X,d) be a g.m.s. with continuous generalized metric d and $f : X \to X$ a contractive self mapping. If there exists a point $x_0 \in X$ such that the corresponding sequence of iterates $\{f^n x_0\}$ contains a convergent subsequence $\{f^{n_i} x_0\}$, then $u = \lim\limits_{i\to\infty} f^{n_i} x_0$ is a unique fixed point of f.*

A mapping f of a g.m.s. X into itself is said to be ε-contractive if there exists $\varepsilon > 0$ such that

$$0 < d(x,y) < \varepsilon \text{ implies } d(fx,fy) < d(x,y).$$

Proposition 4.3. *Let (X,d) be a g.m.s. with continuous generalized metric d and $f : X \to X$ an ε-contractive mapping. If for some $x \in X$, the sequence of iterates $\{f^n x\}$ has a subsequence $f^{n_i} x \to u \in X$, then u is a periodic point, that is, there exists a positive integer k such that $f^k u = u$.*

Also, a Suzuki-Edelstein-type result can be easily obtained under some additional assumptions.

Proposition 4.4. *Let (X,d) be a compact g.m.s. with continuous generalized metric d and let $T : X \to X$ be a continuous mapping. Assume that, for all $x,y \in X$,*

$$\frac{1}{2} d(x,Tx) < d(x,y) \textit{ implies } d(Tx,Ty) < d(x,y).$$

Then T has a unique fixed point.

It is an open question whether continuity of d in Propositions 4.1–4.4 and continuity of T in Proposition 4.4 can be omitted.

4.3. **Multivalued mappings in g.m.s.** It is well-known that there are a lot of fixed point results for multivalued mappings in metric spaces. Most of them are based on contractive conditions that use Hausdorff-Pompeiu metric, defined on the family $CB(X)$ of bounded and closed subsets of a metric space (X,d) by the formula

$$\mathcal{H}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, \tag{7}$$

for $A,B \in CB(X)$. $\mathcal{H}$ is a metric on $CB(X)$ and the metric space $(CB(X),\mathcal{H})$ is complete if and only if (X,d) is complete.

However, an analogous construction is not possible in g.m. spaces, as the following easy example shows.

Example 4.1. Let $X = \{a,b,c\}$ and let $d : X \times X \to [0,+\infty)$ be defined by $d(a,b) = 4$, $d(a,c) = d(b,c) = 1$, and $d(x,x) = 0$, $d(x,y) = d(y,x)$ for all $x,y \in X$. The rectangular inequality (iii) has to be checked only in the case when $x = y$, when it becomes trivial. Hence, (X,d) is a g.m.s., which obviously is not a metric space.

Let $\mathcal{H}$ be defined by (7), and consider the quadrilateral $(\{a\},\{b\},\{a,c\},\{c\})$, with d-closed and d-bounded vertices. It is easy to see that

$$\mathcal{H}(\{a\},\{b\}) = 4 > 1+1+1 = \mathcal{H}(\{a\},\{a,c\}) + \mathcal{H}(\{a,c\},\{c\}) + \mathcal{H}(\{c\},\{b\}).$$

Hence, rectangular inequality is not satisfied, and $(CB(X),\mathcal{H})$ is not a g.m.s.

Thus, in order to obtain multivalued fixed point results in g.m.s., possibly another definition of a metric on $CB(X)$ is needed.

4.4. **Coupled fixed points in g.m.s.** Finally, we recall that a lot of coupled fixed point results has been obtained recently for mappings with two variables in (ordered) metric spaces. Here, for a mapping $F : X \times X \to X$, a pair $(a, b) \in X^2$ is called a coupled fixed point if $F(a, b) = a$ and $F(b, a) = b$ hold. It was shown (see, e.g., [1]) that, in most cases, these results can be deduced from certain known results for mappings with one variable, using the well-known fact that, for the given metric space (X, d), the following formulas

$$\begin{aligned} d_+((x, y), (u, v)) &= d(x, u) + d(y, v), \\ d_{\max}((x, y), (u, v)) &= \max\{d(x, u), d(y, v)\} \end{aligned}$$

define metrics on the set $X \times X$. The following simple example shows that similar constructions are not possible in g.m.s.

Example 4.2. Consider the g.m.s. (X, d) defined in Example 4.1, and the quadrilateral $((a, b), (b, c), (a, c), (c, c))$ in $X \times X$. Then

$$\begin{aligned} d_+((a, b), (b, c)) &= 5 > 1 + 1 + 1 = \\ &= d_+((a, b), (a, c)) + d_+((a, c), (c, c)) + d_+((c, c), (b, c)) \end{aligned}$$

and

$$\begin{aligned} d_{\max}((a, b), (b, c)) &= 4 > 1 + 1 + 1 = \\ &= d_{\max}((a, b), (a, c)) + d_{\max}((a, c), (c, c)) + d_{\max}((c, c), (b, c)). \end{aligned}$$

Hence, in both cases, rectangular inequality is not satisfied and (X^2, d_+) and $(X^2, d_{\max})$ are not g.m.s.

5. Conclusion

Fixed point theory in metric spaces have many applications. It is natural that there have been several attempts to extend it to a more general setting. One of these generalizations was introduced by Branciari in 2000, where the triangle inequality was replaced by a so-called quadrilateral inequality. It was not immediately observed that such spaces (called generalized metric spaces) may fail to satisfy some standard metric properties. Hence, in some of the first papers that followed, the authors implicitly used some of these additional conditions.

We show in this article that, nevertheless, most of these results are valid, since their proofs can be corrected, using some easy observations due to Turinici and other authors. Moreover, all the results of authors who, later on, assumed some additional requirements (as, e.g., Hausdorffness of the respective topology) can be made more general, by omitting these assumptions.

Some open questions and suggestions for further work are also noted.

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